UDC 536.212

ASYMMETRICAL R- AND RC-NETWORK ELECTRICAL MODELS FOR SOLVING NONLINEAR EQUATIONS OF UNSTEADY HEAT CONDUCTION

L. A. Kozdoba, L. V. Knyazev, and I. D. Konoplev

Inzhenerno-Fizicheskii Zhurnal, Vol. 8, No. 5, pp. 667-674, 1965

A method is given for calculating the parameters of R- and RC-network electrical models for solving nonlinear equations of unsteady heat conduction with variable heat sources (sinks) and given variable boundary conditions of the first to fourth kinds.

In solving equations of unsteady heat conduction on electrical models with R and RC networks, the body is divided by planes into elementary volumes in the form of irregular triangular pyramids. This allows greater accuracy in approximating the body shape than rectangular division,* and greater accuracy in assigning the boundary conditions. The use of asymmetrical networks in combination with rectangles allows free variation of the space intervals, i.e., simplifies the choice of transition zones. The transition zones usually employed are parts of the network where the elementary areas are right triangles.

A method of calculating transition zones for solving steady-state problems on R-networks was given in [3]; in [4] it was shown that the transition zones of [3] may be used to solve unsteady problems on R-networks by the method of [5].

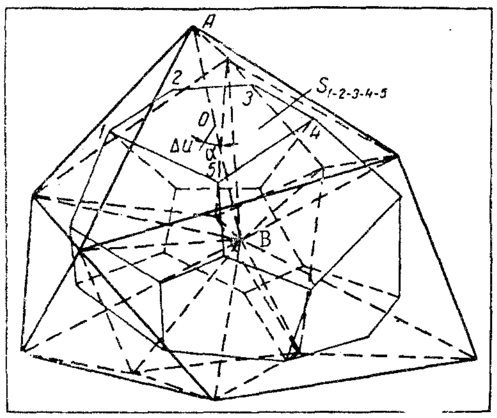

Fig. 1. Part of the body divided into elementary volumes: For an RC network, the currents modeling w and capacitances are supplied at A, B, ... [1, 2]. For an R-network, resistances R_w, R_t are connected at A, B, ... [5, 8, 10]. For both R-C and R-networks, capacitances or resistances modeling the boundary conditions are connected to the surface nodes.

Asymmetrical triangular networks for solving two-dimensional steady-state problems were proposed in [6].

The nonlinear equation for the potential u has the form

$$\nabla(\mu\nabla u) + \beta - \gamma\,\partial u/\partial i = 0. \tag{1}$$

The boundary conditions are

$$u_s = f(x,\ y,\ z), \tag{2}$$

$$\Theta(t) = -\mu(\partial u/\partial n)_s, \tag{3}$$

$$\delta[u_s(t) - u_m(t)] = -\mu(\partial u/\partial n)_s, \tag{4}$$

$$u_s(t) = u_m(t) \quad \text{or} \quad -\mu_s(\partial u/\partial n)_s = -\mu_m(\partial u_m/\partial n)_s. \tag{5}$$

The initial condition is

$$u(x,\ y,\ z,\ 0) = f_1(x,\ y,\ z). \tag{6}$$

In the solution of the equations of unsteady heat conduction the values of (2)-(5) are called boundary conditions of the I-IV kinds [7].

Figure 1 shows part of a body divided into elementary volumes in the form of irregular triangular pyramids, the angles of the triangular faces being not more than 90°. The heavy lines correspond to the discrete resistances of an asymmetrical network replacing this part of the body and solving the equation of the potential u. The resistances R_t, R_w or R_α, connected to the nodes when the solution is based on an R-network, and the capacitances and R_α, connected to the nodes when the solution is based on an RC network, have been omitted.

The light lines are the result of the intersection of planes normal to the edges of faces of the pyramid, whose vertices are located at the point B.

The distribution of potential u along the i-th resistance (for example, AB) at time n may be interpreted as a linear integral between points A and B

$$u_{B,n} - u_{A,n} = \int_A^B \nabla u\, dl. \tag{7}$$

*We have in mind the solution of an equation written in a rectangular coordinate system; the situation is similar for a cylindrical or spherical coordinate system.

The current in the resistance may be interpreted as the total normal flux cutting the boundary of the volumes surrounding both points and bounded by the polygons indicated by the light lines in Fig. 1.

Since the current density is equal to $-\mu\nabla u$, we obtain

$$J_{AB} = -\iint_S \mu (\nabla u \cdot j)\, dS, \tag{8}$$

where j is the unit vector normal to dS.

If ∇u and $\mu\nabla u$ are expanded in a Taylor series relative to the point bisecting AB, and all the terms besides the first are neglected, then $\nabla u \approx (\nabla u)_0$, and $\mu\nabla u \simeq \mu_0(\nabla u)_0$.

Since AB is normal to the area $S_{1-2-3-4-5}$, the projection of ∇u on dl is the same as the projection of ∇u on j. Therefore

$$u_{B,\,n} - u_{A,\,n} \simeq l_{AB} |\nabla u|_0 \cos\alpha, \tag{9}$$

$$J_{AB} \simeq -\mu_0 S_{1-2-3-4-5} |\nabla u|_0 \cos\alpha. \tag{10}$$

The resistance connecting A and B is equal to

$$R_{AB} = \frac{u_{A,\,n} - u_{B,\,n}}{J_{AB}} = \frac{l_{AB}}{\mu_0 S_{1-2-3-4-5}}, \tag{11}$$

i.e., R_{AB} depends on the physical properties of the material and the AB method of dividing up the volume. If $S_{1-2-3-4-5}$ were not normal to AB, the value of R_{AB} would depend on the orientation of the field.

It is necessary to determine the volumes relating to the terms β and $\eta \frac{\partial u}{\partial t}$ in (1). If (1) is integrated over the volume surrounding the point B (Fig. 1), we obtain

$$\iiint_v \nabla(\mu\nabla u)\, dv + \iiint_v \beta\, dv - \iiint_v \eta \frac{\partial u}{\partial t}\, dv = 0. \tag{12}$$

Or, if $\partial u/\partial t$ is written in finite differences,

$$\iiint_v \nabla(\mu\nabla u)\, dv + \iiint_v \beta\, dv + \iiint_v \eta \frac{u_{B,\,n-1} - u_{B,\,n}}{\Delta t}\, dv = 0. \tag{13}$$

Then, by the Ostrogradskii-Gauss divergence theorem,

$$\iiint_v \nabla(\mu\nabla u)\, dv = \iint_S \nabla(\mu\nabla u \cdot j)\, dS. \tag{14}$$

There are no other terms on the right, since the corresponding unit vectors reduce them to zero.

Substituting (14) into (12) and (13), we obtain

$$\iint_S \nabla(\mu\nabla u \cdot j)\, dS + \iiint_v \beta\, dv - \iiint_v \eta \frac{\partial u}{\partial t}\, dv = 0, \tag{15}$$

$$\iint_S \nabla(\mu\nabla u \cdot j)\, dS + \iiint_v \beta\, dv + \iiint_v \eta \frac{u_{B,\,n-1} - u_{B,\,n}}{\Delta t}\, dv = 0. \tag{16}$$

If the volume integrals in (15) and (16) are replaced by the values β, $\eta \frac{\partial u}{\partial t}$ and $\eta \frac{u_{B,\,n-1} - u_{B,\,n}}{\Delta t}$, measured at B and multiplied by the volume bounded by areas perpendicular to sides iB (for example, $S_{1-2-3-4-5}$ perpendicular to AB), while the area integral is replaced by the currents in the network* from Eq. (8), we obtain, respectively,

$$\sum_i J_{iB} + \beta_B v - \eta_B \left(\frac{\partial u}{\partial t}\right)_B v = 0, \tag{17}$$

$$\sum_i J_{iB} + \beta_B v + \eta_B \frac{u_{B,\,n-1} - u_{B,\,n}}{\Delta t} v = 0. \tag{18}$$

*As long as we are dealing with a network of discrete resistances, i.e., of resistances to the flow of potential u, with the currents of this potential, and so forth.

Substituting (11) into (17) and (18), we obtain the generalized finite-difference equation of potential u for B:

$$\sum_i \mu_0 \left(\frac{S_i}{l_{iB}}\right)(u_{i,n} - u_{B,n}) + \beta_B v - \eta_B \left(\frac{\partial u}{\partial t}\right)_B v = 0, \tag{19}$$

$$\sum_i \mu_0 \left(\frac{S_i}{l_{iB}}\right)(u_{i,n} - u_{B,n}) + \beta_B v + \eta_B \frac{u_{B,n-1} - u_{B,n}}{\Delta t} v = 0. \tag{20}$$

This method may be used in a network of any configuration in which the normals to the center of the branch converge in one point. Accordingly, besides triangles, the method may be applied to rectangles, regular hexagons, and isosceles trapezia.

MacNeal [6] examined the potential equation for a two-dimensional steady-state problem in this way, stipulating that the perpendiculars to the sides need not necessarily bisect them, but must converge in one point.

For the electric potential (19) and (20) are written in the form

$$\sum_i \frac{V_i - V_B}{R_{ei}} + I_B - C_B \frac{\partial V}{\partial t_e} = 0, \tag{21}$$

$$\sum_i \frac{V_i - V_B}{R_{ei}} + \frac{V_M - V_{B,n}}{R_{WB}} + \frac{V_{B,n-1} - V_{B,n}}{R_t} = 0. \tag{22}$$

For the thermal potential (19) and (20) may be written

$$\sum_i \frac{T_{i,n} - T_{B,n}}{R_{\tau i}} + w_B v_t - (c\gamma)_{B,n-1} \left(\frac{\partial T}{\partial t_t}\right) v_t = 0, \tag{23}$$

$$\sum_i \frac{T_{i,n} - T_{B,n}}{R_{ti}} + w_B v_t + (c\gamma)_{B,n-1} \frac{T_{B,n-1} - T_{B,n}}{\Delta t} v_t = 0. \tag{24}$$

The analogy requires that the parameters of the RC network solving (23) be equal to

$$R_{ei} = l_{tBi} R_N / \lambda_0 S_{ti}, \tag{25}$$

$$I_B = w_B v_t / K R_N, \tag{26}$$

$$C_B = (c\gamma)_{B,n-1} v_t \tau / R_N \tag{27}$$

or the time scale to

$$\tau = t_e / t_t = R_{ei} C_B / R_{ti} (c\gamma) v_t. \tag{28}$$

The parameters of the R-network solving (24) are

$$R_{ei} = l_{tBi} R_N / \lambda_0 S_{ti}, \tag{29}$$

$$R_W = V_M K R_N / w v_t \tag{30}$$

with $V_M \gg V_{B,n}$ if w is a heat source, or with $V_M \ll V_{B,n}$ if w is a heat sink; as in [10-12],

$$R_t = \Delta t R_N / (c\gamma)_{B,n-1} v_t. \tag{31}$$

Expressions for the parameters of the RC and R-networks, used in specifying the boundary conditions, are derived analogously, as in [1, 2, 5, 8, 9, 10, 12, 13].

The method presented above may be used successfully not only to solve unsteady heat conducting problems on pure network models RC and R), but also on combined models: RC continuum [11] or R-continuum [12]*.

An example of the determination of the unsteady temperature field of a plate using an R-network with asymmetrical unequal cells is given below.

The results are compared with the solution for an R-network and for a combined model (R-network and conductive paper). The nodes of the R-network and the combined model are located at the corners of squares. Results of solutions on R-networks and combined models of this kind have been frequently compared [4-6, 8-14, etc.] with analytical (exact and approximate) and numerical solutions and with the data of thermal experiments on models and full-scale

*The combined model in [12] is an R-network and conductive paper.

systems. Therefore, solutions obtained on R-networks and combined models (square cells) may serve as a reference for checking the validity of solutions on asymmetrical networks.

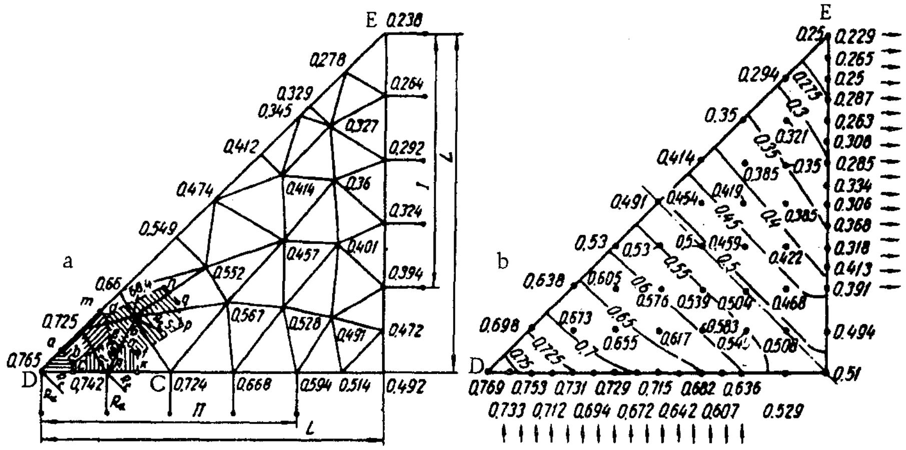

Fig. 2. Asymmetrical network (a) and nodes of network with square cells (b). The temperature distribution is given (as fractions of T) for the steady-state regime.

First, optimal time and space intervals were chosen for solving the control problem on a network with square cells.

Since the unsteady finite-difference heat conduction equation is solved on R-networks or combined models (continuum and R-network) is using an implicit scheme, the relationship of the space and time intervals affects only the accuracy, not the stability or convergence of the solution.

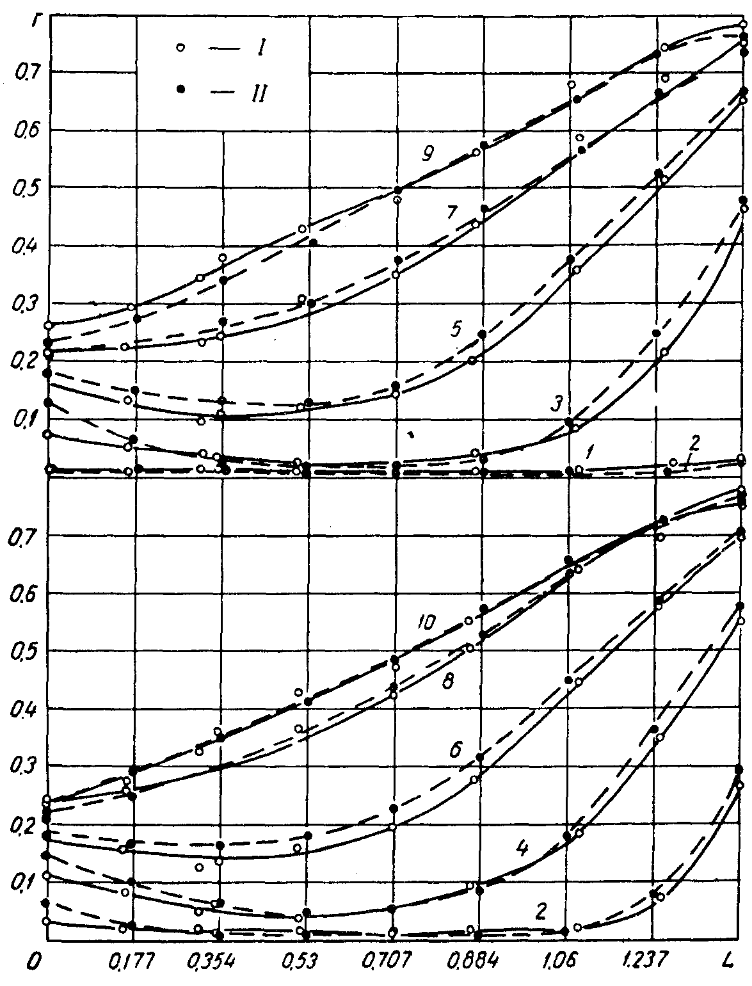

Fig. 3. Temperature distribution with time along diagonal DE (Fig. 2). I and II) results of solution on the asymmetrical network and on the network with square cells and combined model: 1) 5 sec, 2) 150, 3) 500, 4) 1000, 5) 2500, 6) 3500, 7) 7500, 8) 11 500, 9) 23 500 sec, 10) steady-state conditions.

Figure 2a shows one-half of the cross-section of a square plate (of side L), for which boundary conditions of third kind are specified on sections I, II of length 0.75L:

$$\mathrm{Bi} = \alpha L/\lambda = 10.$$

The temperature of the medium in sections I, II is, respectively, 0.2T and 0.8T. The initial temperature of the plate is zero.

Figure 2a shows an asymmetrical triangular network (resistances R_t are shown only for the three points A, B, D) and the steady-state temperatures at the nodes.

Figure 2b shows the square network on which the control problem is solved; the temperatures are those obtained on the square network and the isotherms those obtained on the combined model under steady-state conditions.

The values of the parameters of the asymmetrical network (Fig. 2a) were calculated from expressions derived from (29)-(31), taking into account the remark about calculating the parameters of the network for modeling the boundary conditions. For example (Fig. 2a),

$$R_{\mathrm{AB}} = \lambda_0 \frac{l_{\mathrm{AB}}}{l_{ef}} R_N ; \; R_{\mathrm{AB}} = \lambda_k \frac{l_{\mathrm{AB}}}{l_{fk}} R_N ;$$

$$R_{t\mathrm{A}} = \frac{\Delta t}{(c\gamma)_{\mathrm{A},\, n} S_{cbefk}} R_N ; \quad R_{t\mathrm{B}} = \frac{\Delta t}{(c\gamma)_{\mathrm{B},\, n} S_{defpqnd}} R_N ;$$

$$R_{\alpha\mathrm{A}} = R_N / \alpha_{A,\, n} l_{ck} \quad \text{etc.}$$

If boundary conditions of the second kind had been specified at the corresponding part of the surface, then $R_{q\mathrm{A}} = V_{\mathrm{M}} K R_N / q_{\mathrm{A},\, n} l_{ck}$.

In this case, as always, conditions assumed in deriving R_q are the same as in deriving (30).

Figure 3 shows the temperature distribution with time along diagonal DE of Fig. 2 for various values of time, obtained on the asymmetrical network, and the network with square cells and the combined model. The results of solving on the last two models practically coincide.

Therefore, the method proposed may be used to calculate the R and R-C parameters of pure network and combined models with asymmetrical division.

An asymmetrical network degenerates into a rectangular one, but the method of calculation does not change.

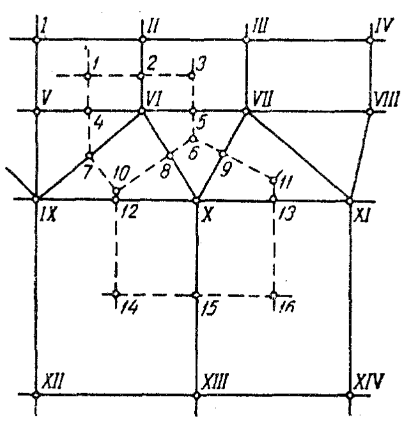

Fig. 4. Network with transition zone.

Figure 4 shows a rectangular network with cells of different sizes and a transition zone. The resistances in the regions with rectangular cells of different sizes and in the transition zone are calculated by the general method. For example,

$$R_{\mathrm{X-VI}} = \lambda_8 \frac{l_{\mathrm{X-VI}}}{l_{10-6}} R_N , \qquad R_{\mathrm{X-VI}} = \lambda_7 \frac{l_{\mathrm{X-VI}}}{l_{7-10}} R_N ,$$

$$R_{V-\mathrm{VI}} = \lambda_4 \frac{l_{V-\mathrm{VI}}}{l_{1-7}} R_N , \qquad R_{t\mathrm{VI}} = \frac{\Delta t}{(c\gamma)_{\mathrm{VI},\, n} S_{1-3-6-10-7-1}} R_N, \quad \text{etc.}$$

Similar expressions for the parameters of R and R-C networks may be derived for the dimensionless equation of unsteady heat conduction and the corresponding boundary conditions, as in [13], and for systems of heat and mass transfer equations, as in [14].

NOTATION

R – ohmic resistance; C – capacitance; u – potential; μ, η, δ – coefficients in (1)-(5); β and Θ – internal and external sources, respectively; t – time; S – area; Δt – time interval; v – volume; V – electrical potential; T – temperature; $R_N = R_{ei}/R_{ti}$ – scale factor for conversion from thermal to electrical resistances; I – electric current; $K = \frac{T - T_{\min}}{V - V_{\min}}$ – scale factor for conversion from temperatures to voltages; w – heat source or sink function; q – specific heat flux per unit area and unit time; α, λ, c – heat transfer coefficient, thermal conductivity and volume specific heat, respectively; R_α, R_q, R_w, R_t – resistances, parameters of R-network as in [4, 5, 8, 10, 12, 13]. Indices: s – surface; m – medium; n – number of time interval $t = n_1 \Delta t_1 + n_2 \Delta t_2 + \ldots$; A, B – relate quantities to the points A, B, ..., AB – relate quantities to the segment AB; e – electrical model; t – thermal model.

REFERENCES

1. L. I. Gutenmakher, Electrical Models [in Russian], Izd-vo AN SSSR, 1949.
2. I. M. Tetel'baum, Electrical Modeling [in Russian], Fizmatgiz, 1959.
3. L. V. Nitsetskii, Nauchnye doklady vysshei shkoly, elektronekhanika i automatika, no. 2, 1956.
4. L. A. Kozdoba and V. I. Makhnenko, Collection No. 1 of the Proceedings of Fourth Conference on the Use of Physical and Mathematical Models [in Russian], Izd-vo MEI, 1962.
5. G. Liebmann, Trans. ASME, 78, no. 3, 1956.
6. R. H. MacNeal, Quart. Appl. Math., 11, 295, 1953.
7. A. V. Lykov, Theory of Heat Conduction [in Russian], GITTL, 1952.
8. G. Liebmann, Trans. ASME, 78, no. 6, 1956.
9. L. A. Kozdoba, Izv. VUZ. Energetika, no. 2, 1960.
10. L. A. Kozdoba, IFZh, no. 7, 1960.
11. V. Paschkis, Trans. ASME, 81, C, no. 2. 1959.
12. L. A. Kozdoba and V. I. Makhnenko, collection: Heat and Mass Transfer [in Russian], V, Izd-vo AN BSSR, Minsk, 1963.
13. L. A. Kozdoba and V. I. Makhnenko, Izv. VUZ. Energetika, no. 6, 1962.
14. L. A. Kozdoba, IFZh, no. 2, 1963.

1 June 1964

Institute of Naval Engineers,
Odessa